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Stability of Distributed Systems With Feedback via Michailov's Criterion

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This paper is based on results derived during a stability study of the Saturn V rocket for which it was necessary to validate the use of Nyquist's encirclement-counting technique in distributed systems. An outline of the paper is as follows: Certain results concerning the finiteness of the number of zeros of polynomials in s and e^s are shown in Theorem 1 and its corollaries. Theorem 2 is a generalization of Michailov's Criterion. Simplifying assumptions, usually valid in practice, yield a simplified test to determine if "encirclement-counting" is a valid stability test [*equation (20)*]. *The results are reformulated for an open-loop analysis. Various aspects of the theory are shown by three examples based on an electrical equivalent of a simple single-engine, liquid-fuel rocket.*

I. INTRODUCTION

Liquid-fueled rockets can exhibit a peculiar type of instability due to self-sustained longitudinal oscillations. Since the rocket then stretches and shrinks longitudinally, it behaves like a pogo-stick, which has resulted in the nickname POGO for this type of instability.

To see how this phenomenon arises, consider the simple diagram shown in Fig. 1. The chain of events which can cause POGO is initiated by a random variation in thrust of the engine. This thrust variation causes the rocket structure to oscillate in its natural modes. The pressure in the fuel tank thus varies. This pressure variation is propagated down the fuel feed line, resulting in a variation of fuel flow into the engine. Since the thrust of the engine is proportional to the rate of fuel entering, the loop is completed, and instability results if this resulting thrust variation aids the original random variation which initiated the chain of events.

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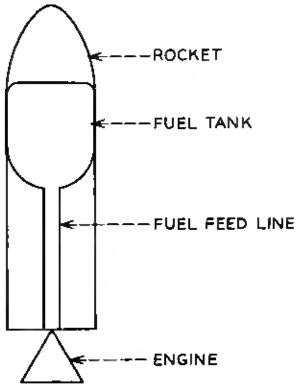

Fig. 1—Simple model of a liquid-fuel rocket.

In modeling this physical situation, it is customary to perform a modal analysis of the rocket structure, and retain only the most significant modes. This results in the distributed structure being replaced by a lumped approximation. Although the same technique could be used to lump the feed-line, there are many reasons for desiring to keep this element as a distributed parameter. There is no great difficulty in doing so, since the fluid equations which govern the feed-line are of the same form as electrical transmission lines.

An equivalent circuit of Fig. 1 would then be Fig. 2, where $V_1(s)$, $V_2(s)$ are the Laplace transforms of the pressure variations at the top and bottom of the feed-line, respectively, and $I_1(s)$, $I_2(s)$ represent the transform of flow variations. $E_1(s)$ is then the random pressure variation at the top of the line due to the assumed random thrust variation above. $Y(s)$ represents the hydro-mechanical impedance of the feed-line output, and $G(s)$ includes the structural feedback. To see the type of equations which will be of concern for a stability analysis, assume that losses can be neglected in the feed-line.

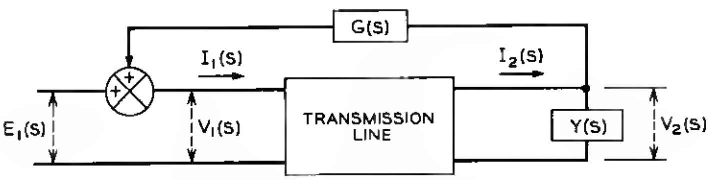

Fig. 2—Electrical equivalent of Fig. 1.

The system equations can be written in matrix form as:

$$\begin{bmatrix} \cosh ks & -Z_c \sinh ks & -1 & 0 \\ -\dfrac{1}{Z_c}\sinh ks & \cosh ks & 0 & -1 \\ -1 & 0 & G(s) & 0 \\ 0 & 0 & Y(s) & -1 \end{bmatrix} \begin{bmatrix} V_1(s) \\ I_1(s) \\ V_2(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -E_1(s) \\ 0 \end{bmatrix} \tag{1}$$

where Z_c (a positive real number) is the characteristic impedance of the line and k (a positive real number) is the ratio of line length to wave speed in the line. The determinant ($\Delta(s)$) of the matrix appearing in (1) is of interest in a stability analysis. It can easily be computed to be

$$\Delta(s) = \cosh ks - G(s) + Z_c Y(s) \sinh ks. \tag{2}$$

In general, neither $Y(s)$ nor $G(s)$ need be rational (especially in the case of multiple engines) but for simplicity, suppose that

$$G(s) = \frac{N_G(s)}{D_G(s)} \qquad Z_c Y(s) = \frac{N_Y(s)}{D_Y(s)} \tag{3}$$

where N_G, D_G, N_Y, D_Y are polynomials in the complex variable s. Then (2) can be written as

$$\Delta(s)[D_G(s)D_Y(s)] = [D_G(s)D_Y(s)] \cosh ks - N_G(s)D_Y(s) + D_G(s)N_Y(s) \sinh ks. \tag{4}$$

In the following sections of this paper we will be concerned with polynomials in the two complex variable s and e^s. We now show that (4) can be considered as such. Thus multiply (4) by $2e^{-2ks}$ to give

$$F(s) = 2e^{-ks}\, \Delta(s)[D_G(s)\, D_Y(s)] = D_G(s)[D_Y(s) + N_Y(s)] - 2N_G(s)\, D_Y(s)e^{-ks} + D_G(s)[D_Y(s) - N_Y(s)]e^{-2ks} \tag{5}$$

or

$$F(s) = \sum_{i=0}^{2} R_i(s)e^{-iks} \tag{6}$$

where each $R_i(s)$ is a polynomial in s.

This paper will also be concerned with open-loop and closed-loop expressions, and will assume that the quantities of interest will be of the form of (6). To show that this is true in our present example, assume that $G(s)$ "closes the loop" and solve for the "line transfer function"

$V_2(s)/V_1(s)$. The result is

$$\frac{V_2(s)}{V_1(s)} = \frac{1}{\cosh ks + Z_C Y(s) \sinh ks}. \tag{7}$$

Hence the "loop gain," $G_0(s)$, is simply $G(s)$ times (7), or

$$\begin{aligned} G_0(s) &= \frac{G(s)}{\cosh ks + Z_C Y(s) \sinh ks} \\ &= \frac{D_Y(s) N_G(s)}{D_Y(s)\, D_G(s) \cosh ks + D_G(s) N_Y(s) \sinh ks}. \end{aligned} \tag{8}$$

The denominator of (8) is seen to be of the form of $\Delta(s)$, and hence can be made to look like (6).

Finally, we wish to remove the simplifying assumption that $Y(s)$ and $G(s)$ are rational. This is desired since an actual engine is not only fed liquid fuel, but also liquid oxidizer. Thus even a single-engine rocket has two feed-lines. (The Saturn V has five engines, for a total of ten feed-lines.) The structure of a model for rockets of this complexity would be that of Fig. 2, repeated once for each feed-line, with suitable interconnections through lumped (i.e., rational) transfer functions. Thus the $G(s)$ and $Y(s)$ of Fig. 2 will be the ratio of sums of powers of s and e^s. As such they can be combined to yield forms such as (6).

A final constraint on the stability analysis is that it is required that the analysis be of the conventional open-loop type using Nyquist's Criterion. For lumped systems this presents little difficulty since one can always make open-loop measurements at as high a frequency as necessary to guarantee that all singularities of the transfer function are included. Furthermore, for rational functions, no difficulty is encountered in closing the contour in the right-half s-plane. With distributed systems, however, it is possible that the gain becomes periodic for large magnitude of s, and care must be exercised in determining closed-loop stability via open-loop gain plots. What is desired, therefore, is a set of conditions under which the conventional "encirclement counting" technique for lumped systems, remains valid for distributed systems of the type described.

There are two important techniques for determining whether a polynomial in the two complex variables s and e^s has any zeros for $\text{Re}\,[s] \geqq 0$. These are the Pontryagin Criterion[1] and the Michailov Criterion.[2] It is of interest to see if these criteria can be applied to the ratio of such polynomials in order to determine stability of closed-loop gain. In a previous note[3] it was shown that this is not feasible for criteria

of the Pontryagin type. In this paper we are able to develop a stability criterion of the Nyquist type from Michailov's Criterion for a large class of distributed-parameter systems, in particular, for a large class of transmission line systems with feedback.

An outline of the paper is as follows: Certain results concerning the finiteness of the number of zeros of polynomials in s and e^s are stated as Theorem 1 and its corollaries. Theorem 2 is the desired generalization of Michailov's Criterion. Simplifying assumptions, usually valid in practice, yield a simplified test to determine if "encirclement-counting" is a valid stability test [equation (20)]. The results are reformulated for an open-loop analysis. Three examples show various aspects of the theoretical analysis. The Appendix includes a statement of Pontryagin's Criterion suitable for use in the present paper. Also included are the proofs of the two theorems and a derivation of some conditions under which Michailov's Criterion can be simplified.

II. MICHAILOV'S CRITERION

As a starting point we consider an equation of the form

$$G(z) = \sum_{i=1}^{m} \sum_{j=0}^{n} \bar{a}_{ij} z^j e^{\omega_i z} = 0 \tag{9}$$

where $\bar{a}_{ij}$ are complex and ω_i are real. If any of the ω_i were negative, we could multiply $G(z)$ by $e^{|\omega_k| z}$, where ω_k is the most negative of the ωs. This would not change the zeros of $G(z)$, so we assume $0 \leqq \omega_1 < \omega_2 < \cdots < \omega_m$. Dividing by $e^{\omega_m z}$ and letting $\bar{a}_{m-i+1,j} = a_{ij}$, we transform (9) into

$$F(z) = \sum_{i=1}^{m} \sum_{j=0}^{n} a_{ij} z^j e^{-r_i z} \tag{10}$$

where $r_i = \omega_m - \omega_{m-i+1} > 0$ for $i = 2, 3, \cdots, m$ and $r_1 = 0$. [To relate this to the Pontryagin Criterion, note that if the ω_i are rational (which can always be assumed in a practical situation) then a suitable scaling of the z variable will make $G(z)$ of (9) into a polynomial like $H(z)$ of the Pontryagin Criterion.]*

Before continuing, it should be noted that a proof of the Michailov Criterion for exponential polynomials has been presented in the literature.[2] However, this proof assumed that

$$|a_{1n}| > \sum_{i=2}^{m} |a_{in}|. \tag{11}$$

* See Appendix and Ref. 1.

As will be seen by the example to be considered later, in transmission line systems, (11) is almost never satisfied. Hence, a proof is desired which is free of this assumption. However, we do use the assumption that $a_{1n} \neq 0$. That this is no loss of generality can be seen by the following considerations. It is clear that $|r_m| \geqq |r_i|$, $i = 2, \cdots, m$. Then multiplying (10) by $e^{r_m z}$ (which does not change the zeros) puts (10) into the Pontryagin form. If a_{1n} is zero, the principal term is missing and we are finished with the stability study.* To aid in the subsequent development, let us rewrite (10) as

$$F(z) = \sum_{j=0}^{n} z^j Q_j(e^{-z}) \tag{12}$$

where $Q_j(e^{-z}) = \sum_{i=1}^{m} a_{ij} e^{-r_i z}$, or as

$$F(z) = z^k F_k(z) + \sum_{j=0}^{k-1} z^j Q_j(e^{-z}) \tag{13}$$

where k is an integer between zero and n and

$$F_k(z) = \sum_{j=k}^{n} z^{j-k} Q_j(e^{-z}). \tag{14}$$

In the Appendix we prove the following theorem:

Theorem 1: If there exists a non-negative integer $k \leqq n$ such that $F_k(z)$ of (14) has at most a finite number of zeros on Re $(z) \geqq 0$, *then $F(z)$ of (13) has at most a finite number of zeros on* Re $(z) > 0$.

The following corollaries are of interest:

Corollary 1: If there exists a non-negative integer $k \leqq n$ such that $F_k(z)$ of (14) has no zeros on Re $(z) \geqq 0$, *then $F(z)$ of (12) has at most a finite number of zeros on* Re $(z) > 0$.

Corollary 2: If $Q_n(e^{-z})$ of (12) has no zeros on Re $(z) \geqq 0$ *then $F(z)$ of (12) has at most a finite number of zeros on* Re $(z) > 0$.

[Corollary 2 follows since $Q_n(e^{-z}) = F_n(z)$.]

Let

$$F(z) = 1 + \psi(z) z^k F_k(z) \tag{15}$$

where

$$\psi(z) = \sum_{j=0}^{k-1} z^{(j-k)} \frac{Q_j(e^{-z})}{F_k(z)}. \tag{16}$$

In the Appendix we derive theorem 2:

* See Appendix and Ref. 1.

Theorem 2: The number of zeros of $F(z)$ with positive real part is

$$N = \frac{k}{2} - \frac{1}{2\pi}\,\Delta_{-\omega}(F(z)) + \frac{\Delta_C}{2\pi}\,(F_k(z))$$

$$+ \frac{1}{2\pi}\arg(1 + \psi(iy)) - \arg(1 + \psi(-iy)) \tag{17}$$

assuming that $F(z)$ has no purely imaginary zeros and where $\Delta_{-\omega}(F(z))$ is the net change in arg $F(z)$ *along the imaginary axis from $-iy$ to $+iy$, and $\Delta_C(F_k(z))$ is the net change in* arg $(F_k(z))$ *along contuor C, any contour outside the semicircle of radius R of Theorem 1.*

III. MICHAILOV'S CRITERION: SPECIAL CASE

Theorem 2 is the desired statement of Michailov's Criterion. To obtain tighter results, let us now assume that $Q_n(e^{-z})$ has no zeros on Re $(z) \geqq 0$ (i.e., Corollary 2). Further, let the r_i be rational and the a_{ij} be real. By virtue of rational r_i, $Q_k(e^{-z})$ is periodic in y, for $z = x + iy$. Let this period be P. By virtue of real a_{ij}, replacing z by its conjugate results in $Q_k(e^{-z})$ being replaced by its conjugate. Hence we need only consider the semi-infinite strip defined by $x > 0$ and $P \geqq y \geqq 0$.

Michailov's Criterion can be simplified if it can be shown that $Q_n(e^{-z})$ does not wind around the origin as z varies over a suitable C. We now consider this possibility.

$$\begin{aligned} Q_n(e^{-z}) &= \sum_{j=1}^{m} a_{jn} e^{-r_j x}(\cos r_j y - i \sin r_j y) \\ &= a_{1n} + \sum_{j=2}^{m} a_{jn} e^{-r_j x} \cos r_j y - i \sum_{j=2}^{m} a_{jn} e^{-r_j x} \sin r_j y \\ &= \text{Re}\,[Q_n] - i\,\text{Im}\,[Q_n]. \end{aligned} \tag{18}$$

If either Re $[Q_n]$ or Im $[Q_n]$ does not vanish along C, then Q_n cannot wind around the origin. In the Appendix we derive sufficient conditions for this.

We now assume that $\Delta_C[Q_n(e^{-z})] = 0$ and write the Michailov Criterion as

$$N = \frac{n}{2} - \frac{1}{\pi}\,\Delta_{-\varpi/2}(F(z)) + \frac{1}{\pi}\arg(1 + \psi(iy)) \tag{19}$$

where $\Delta_{-\varpi/2}$ is that part of the imaginary axis from 0 to iy. Thus $N = 0$ if

and only if

$$\Delta_{-w/2}(F(z)) = \frac{n\pi}{2} + \arg\,(1 + \psi(iy)). \tag{20}$$

We remark that $\arg\,(1 + \psi(iy))$ can be made close to zero for y sufficiently large.

IV. MICHAILOV'S CRITERION APPLIED TO OPEN-LOOP ANALYSIS

The Michailov Criterion, as well as its predecessor the Pontryagin Criterion, settle the problem of finding rhp zeros of polynomials in z and e^z. In many engineering applications, however, this polynomial is not directly available, but a related ratio of such polynomials can be found. In the study of feedback systems, for example, an open-loop gain can be measured and it is desired to find the poles of the closed-loop gain. These latter poles are the zeros of the polynomial which results from adding the two polynomials whose ratio is the open-loop gain. Stability has been determined for nondistributed systems by counting encirclements of the open-loop gain along the imaginary axis. What we propose to do next is to provide a similar criterion for the distributed parameter problem. Thus let $F(z) = D(z) + N(z)$. Then

$$\begin{aligned}\Delta_\Gamma(D(z) + N(z)) &= \Delta_\Gamma(D(z))\left(1 + \frac{N(z)}{D(z)}\right)\\ &= \Delta_\Gamma(D(z)) + \Delta_\Gamma\left(1 + \frac{N(z)}{D(z)}\right).\end{aligned} \tag{21}$$

Let the contour Γ be composed of a portion of the imaginary axis w and another (possibly semicircular) contour C, such that Γ encloses all zeros of $D(z) + N(z)$. Then

$$\Delta_\Gamma(D(z) + N(z)) = \Delta_\Gamma(D(z)) + \Delta_w\left(1 + \frac{N(z)}{D(z)}\right) + \Delta_C\left(1 + \frac{N(z)}{D(z)}\right). \tag{22}$$

It is the term

$$\Delta_w\left(1 + \frac{N(z)}{D(z)}\right)$$

which is usually available for determining stability. We ask,

$$\text{“When does}\quad \Delta_w\left(1 + \frac{N(z)}{D(z)}\right) = \Delta_\Gamma(D(z) + N(z))?\text{''}.$$

The answer is that this happens exactly when

$$0 = \Delta_\Gamma(D(z)) + \Delta_C\left(1 + \frac{N(z)}{D(z)}\right). \tag{23}$$

To develop a more practical criterion let us rewrite this expression using n_F to be the highest power of z in $F(z)$, and N_F to be the number of zeros of $F(z)$ inside Γ. Then

$$0 = \Delta_\Gamma(D(z)) + \Delta_C(D(z) + N(z)) - \Delta_C(D(z)) \tag{24}$$

$$0 = 2\pi N_D + n_{D+N}\pi - n_D\pi \tag{25}$$

where we have neglected those terms which become small for large z. If we limit further consideration to systems which are open-loop stable (i.e., $N_D = 0$) then (25) requires that $n_{D+N} = n_D$. In most practical situations, the open-loop gain is bounded at infinity, that is to say $n_N \leqq n_D$. Hence $n_{D+N} \leqq n_D$. Since $n_{D+N} < n_D$ requires

$$\lim_{Z \to \infty} \frac{N(z)}{D(z)} = -1,$$

we can conclude that counting encirclements of the open-loop gain is a valid method for determining stability [i.e., (25) is satisfied] for systems which are open-loop stable ($N_D = 0$) and whose gain is bounded at infinity ($n_{D+N} \leqq n_D$) but does not approach -1 for large frequencies ($n_{D+N} \not< n_D$). This includes the case, usually found in practice, that the open-loop gain approaches zero for large frequencies.

V. EXAMPLES

All examples refer to Fig. 2 and are chosen to illustrate various aspects of the analysis. At various points in the examples, the following assumptions concerning $G(s)$ and $Y(s)$ are referred to:

A1. $G(s)$ and $Y(s)$ are each the ratio of two polynomials having real coefficients with no singularities on Re $(s) > 0$.

A2. $\lim_{s \to \infty} G(s) = k_1$ where k_1 is real and $| k_1 | \leqq 1$.

A3. $\lim_{s \to \infty} Y(s) = \lim_{s \to \infty} sC$ where C is non-negative real.

Assumption A1 requires G and Y to be stable transfer functions. Assumption A2 insists that the feedback gain at infinity be less than unity. Assumption A3 is physically appealing.

Some unusual properties of the natural frequencies of this system have been described elsewhere.[4,5]

Example 1: In this example, Assumptions A1, A2, and A3 are invoked, and $G(s)$ and $Y(s)$ are given by (3). We wish to show that assumption

(11), used in previous derivations of Michailov's Criterion, is not met and that it is valid to count encirclements of the Nyquist plot to determine stability. By A1, $D_G(s)$ and $D_Y(s)$ have no right-half plane zeros. Thus $\Delta(s)$ has right-half plane zeros exactly when $\Delta(s)D_G(s)D_Y(s)$ has right-half plane zeros.

From Assumption A2 we conclude that deg $D_G(s) \geqq$ deg $N_G(s)$. From A3 we conclude that deg $N_Y(s) >$ deg $D_Y(s)$. From this, and (5) and (6), we see that the principal term is present and that the assumption (11) used in previous proofs of Michailov's Criterion is not met. In fact, one can readily convince oneself that this will be the case whenever lossless transmission lines are involved, since all exponential terms will involve hyperbolic functions.

Using the notation of (6), the open-loop gain (8) can be expressed as

$$G_0(s) = \frac{R_1(s)e^{-ks}}{R_0(s) + R_2(s)e^{-2ks}} \tag{26}$$

whose norm becomes small for large, right-half plane values of s. Hence it is valid to count encirclements of the open-loop gain about the point $+1$.

Example 2: Here we show the necessity of the open-loop stability requirement. Suppose Assumptions A1 and A2 are invoked and further assume that $D_G(s) = 1$, $N_G(s) = k_1$, $D_Y(s) = 1$, $N_Y(s) = sC_1 + g$. This corresponds to terminating the line in a capacitance C_1 and shunt conductance $g \neq 0$. The open-loop gain for $s = jw$ becomes

$$G_0(jw) = \frac{k_1}{\cos kw - wC_1 \sin kw + ig \sin kw} \tag{27}$$

which is real only when $\sin kw$ is zero. This implies that $\cos kw$ is ± 1. Thus if $| k_1 | < 1$, $G_0(jw)$ cannot encircle the $+1$ point. (This result is in agreement with Assertion 2 of Ref. 5, to which this problem corresponds if $g = 0$. It is intuitive that adding losses to a lossless system will enhance stability.)

To show the necessity of the open-loop stable requirement, note that g can be either positive or negative. From the Pontryagin Criterion, we see that (27) is then stable or unstable, respectively, and that for $| k_1 | < 1$ the closed-loop system is stable or unstable, respectively. However, in either case there are no encirclements of the critical point by the open-loop gain.

Example 3: Let

$$G(s) = \frac{\sum_{i=0}^{n} a_i s^i}{\sum_{j=0}^{n} b_j s^j} \qquad b_n \neq 0$$
$$Z_C Y(s) = \frac{\sum_{k=0}^{p} c_k s^k}{\sum_{l=0}^{p} d_l s^l} \qquad c_p \neq 0. \tag{28}$$

In this final example we look at how the a, b, c, and d coefficients of (28) enter into the $R_i(s)$ polynomials of (6) and into the $Q_n(e^{-z})$ polynomials of Theorem 1 and its corollaries. We show how the Assumptions A2 and A3 affect whether the system satisfies Corollaries 1 and 2, and thereby provide examples of such systems. Using (3) and (28), (5) becomes

$$\begin{aligned} F(s) = s^{n+p}\{ & b_n(d_p + c_p) - 2a_n d_p e^{-ks} + b_n(d_p - c_p)e^{-2ks}\} \\ & + s^{n+p-1}\{[(d_{p-1} + c_{p-1})b_n + b_{n-1}(d_p + c_p) \\ & - 2[a_{n-1}d_p + a_n d_{p-1}]e^{-ks} \\ & + [b_n(d_{p-1} - c_{p-1}) + b_{n-1}(d_p - c_p)]e^{-2ks}\} \\ & + \sum_{m=0}^{n+p-2} s^m \{ \sum_{j+i=m} b_j(d_i + c_i) - 2e^{-ks} \sum_{j+i=m} a_j d_i \\ & + e^{-2ks} \sum_{j+i=m} b_j(d_i - c_i)\}. \end{aligned} \tag{29}$$

First we investigate the zeros of the coefficient of s^{n+p} in (29). [This coefficient corresponds to $Q_n(e^{-z})$ in Corollary 2.) For simplicity let $e^{ks} = z$. This maps the left-half s-plane into the unit circle in the z plane.

If $d_p = c_p$, then the coefficient of s^{n+p} has zeros whenever $b_n c_p = a_n c_p z^{-1}$. If a_n were zero, the coefficient in question would become constant, which satisfies the conditions of Corollary 2. If a_n is not zero, then the condition under discussion simplifies to $z^{-1} = b_n/a_n$. All solutions of this will satisfy $|z| < 1$ if $|a_n| < |b_n|$. Hence all zeros of the coefficient of s^{n+p} in (29) will lie in the left-half plane if $|a_n| < |b_n|$. This is intuitively appealing since this requires that $G(s)$ have less than unity gain at large frequencies (as required by Assumption A2).

On the other hand, if $d_p \neq c_p$, then the zeros of interest are solutions of

$$(z^{-1})^2 - \frac{2a_n d_p}{b_n(d_p - c_p)}(z^{-1}) + \frac{d_p + c_p}{d_p - c_p} = 0. \tag{30}$$

It is well-known[6] that solutions of (30) (for z^{-1}) have magnitude less than unity if and only if the following tbree conditions are met.

$$\left| \frac{d_p + c_p}{d_p - c_p} \right| < 1 \tag{31a}$$

$$1 + \frac{d_p + c_p}{d_p - c_p} = \frac{2d_p}{d_p - c_p} > \frac{2a_n d_p}{b_n(d_p - c_p)} \tag{31b}$$

$$\frac{2d_p}{d_p - c_p} > -\frac{2a_n d_p}{b_n(d_p - c_p)}. \tag{31c}$$

Thus conditions (31) are NAS for $|z| > 1$. Condition (31a) requires that d_p and c_p have opposite sign. Since this corresponds to terminating the line in a negative conductance at high frequencies [i.e., $\lim_{s\to\infty} Y(s) < 0$], we reject this case. If both d_p and c_p are nonzero, and have the same sign, (31a) is violated. If $d_p \neq 0$, (31b) and (31c) together require $|a_n| < |b_n|$ as before. The remaining possibility is that $d_p = 0$. This is a reasonable physical assumption; in fact, it is required by Assumption A3. Invoking Assumptions A2 and A3, the coefficient in question now becomes $b_n c_p(1 - e^{-2ks})$ which has an infinity of purely imaginary zeros, and this example no longer satisfies Corollary 2.

To see if it satisfies Corollary 1, rewrite (29) as

$$\begin{aligned} F(s) = {} & s^{n+p-1}\{(sb_n c_p + b_{n-1}c_p + b_n c_{p-1})(1 - e^{-2ks}) - 2a_n d_{p-1}e^{-ks} \\ & + b_n d_{p-1}(1 + e^{-2ks})\} \\ & + \sum_{m=0}^{n+p-2} s^m \{ \sum_{j+i=m} [b_j(d_i + c_i) - 2e^{-ks}a_j d_i + e^{-2ks}b_j(d_i - c_i)]\}. \end{aligned} \tag{32}$$

We complete this example by finding conditions under which the coefficient of s^{n+p-1} in (32) satisfies the conditions of Corollary 1. This coefficient can be written as

$$\begin{aligned} & ks\frac{b_n c_p}{k} + b_{n-1}c_p + b_n c_{p-1} + b_n d_{p-1} - 2a_n d_{p-1}e^{-ks} \\ & \qquad - e^{-2ks}\left(ks\frac{b_n c_p}{k} + b_{n-1}c_p + b_n c_{p-1} - b_n d_{p-1}\right). \end{aligned} \tag{33}$$

Let $w = ks$. Then (33) becomes

$$\begin{aligned} & \frac{b_n c_p}{k}\left[w + \frac{kb_{n-1}}{b_n} + \frac{kc_{p-1}}{c_p} + \frac{kd_{p-1}}{c_p}\right] \\ & \qquad - 2a_n d_{p-1}e^{-w} - \frac{b_n c_p}{k}\left[w + \frac{kb_{n-1}}{b_n} + \frac{kc_{p-1}}{c_p} - \frac{kd_{p-1}}{c_p}\right]e^{-2w}. \end{aligned} \tag{34}$$

Zeros of (34) are given by the solutions of (35)

$$(w + \alpha) + \gamma e^{-w} - (w + \beta)e^{-2w} = 0 \tag{35}$$

where

$$\alpha = k\left(\frac{b_{n-1}}{b_n} + \frac{c_{p-1}}{c_p} + \frac{d_{p-1}}{c_p}\right)$$

$$\beta = k\left(\frac{b_{n-1}}{b_n} + \frac{c_{p-1}}{c_p} - \frac{d_{p-1}}{c_p}\right)$$

$$\gamma = -2k\,\frac{a_n}{b_n}\,\frac{d_{p-1}}{c_p}.$$

We assume that $\alpha > \beta$ since $\alpha - \beta = 2d_{p-1}/c_p k$ which is positive by Assumption A3.

It is also reasonable to assume $\alpha > 0$, since b_{n-1}/b_n must exceed zero for the denominator of $G(s)$ to be strictly Hurwitz,* and since c_{p-1}/c_p less than zero would imply zeros of $Y(s)$ in the right-half plane. These considerations also imply that $|\alpha| > |\beta|$. Using these assumptions (i.e., $\alpha > \beta$, $\alpha > 0$, $|\alpha| > |\beta|$) it follows that

$$|w + \alpha| > |w + \beta|$$

for Re $(w) \geqq 0$. Evaluating the magnitude of (35) on Re $(w) \geqq 0$ yields

$$\begin{aligned}
&| w + \alpha + \gamma e^{-w} - (w + \beta)e^{-2w} | \\
&\qquad \geqq | w + \alpha | - | \gamma | \, | e^{-w} | - | w + \beta | \, | e^{-2w} | \\
&\qquad > | w + \alpha | - | \gamma | - | w + \beta |.
\end{aligned}$$

If $\lim_{s\to\infty} G(s) = 0$, then $a_n = 0$ and $\gamma = 0$. This means that (35), and hence (33) and (34), have no zeros in the right-half plane and thus Corollary 1 is applicable to this problem.

VI. CONCLUSIONS

It has been shown that the time-honored technique of determining the existence of unstable poles of a closed-loop gain by counting encirclements of the critical point of the open-loop gain along a finite segment of the imaginary axis remains valid for a large class of distributed parameter systems of practical importance for which the open-loop gain approaches zero for large frequencies. Existing limitations of the Michai-

* A well-known necessary condition for a polynomial to be Hurwitz is that all coefficients have the same sign (see, for example, Ref. 6, p. 281).

lov Criterion have been removed so as to include physical systems of lossless transmission lines.

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APPENDIX

A.1 *Pontryagin's Criterion*

Let $h(z, t)$ be a polynomial with complex coefficients in the two complex variables z and t. Pontryagin[1] has developed necessary and sufficient conditions that the function $H(z) = h(z, e^z)$ have zeros with only negative real parts. We now present one of Pontryagin's main results. Let r and s be the degrees of the polynomial $h(z, t)$ with respect to z and t. Then the principal term of $h(z, t)$ is the term containing the product $z^r t^s$. Pontryagin showed that if $h(z, t)$ does not contain the principal term, then $H(z)$ has an infinity of zeros with arbitrarily large positive real parts.

Let $p(\cdot)$ and $q(\cdot)$ be real-valued functions of a real variable. We say that the zeros of these two functions alternate if: (*i*) they have no common zeros, (*ii*) they have only simple zeros, and (*iii*) between every two zeros of one of these functions there exists at least one zero of the other. The result of Ref. 1 which will be used in the present study is:

Let $h(z, t)$ be a polynomial with the principal term and $H(iy) = F(y) + iG(y)$ where $F(y)$ and $G(y)$ take on real values whenever y is real. If all zeros of the function $H(z)$ have negative real parts, then all zeros of $F(y)$ and $G(y)$ are real, alternate, and

$$G'(y)F(y) - F'(y)G(y) > 0$$

where superscript prime denotes the derivative. In order that all zeros of $H(z)$ have negative real parts, it is sufficient that all zeros of $F(y)$ and $G(y)$ are real and alternate and that $G'(y)F(y) - F'(y)G(y)$ be positive for some y.

A.2 *Proof of Theorem 1*

Choose a real number $R' > 1$ such that Re $(z) > 0$ and $F_k(z) = 0$ implies $| z | < R'$. Define the set θ,

$$\theta = \{z \mid \text{Re } (z) > 0 \quad \text{and} \quad | z | > R'.\}.$$

Let

$$D_k = \inf_{z \varepsilon \theta} | F_k(z) | > 0$$

and

$$M_k = \sup_{j=0,\cdots,k-1} \sum_{i=1}^{m} | a_{ij} |.$$

If $M_k = 0$, then $F(z) = F_k(z)$ and the theorem is trivially true. If $M_k > 0$, then for all $z \varepsilon \theta$ the following inequalities hold.

$$\begin{aligned}
| F(z) | &\geqq | z^k F_k(z) | - \sum_{j=0}^{k-1} | z^j Q_j(e^{-z}) | \\
&\geqq | z^k | D_k - \sum_{j=0}^{k-1} | z^j | \sum_{i=1}^{m} | a_{ij} | | e^{-r_i z} | \\
&\geqq D_k | z^k | - \sum_{j=0}^{k-1} | z^j | \sum_{i=1}^{m} | a_{ij} | \\
&\geqq D_k | z |^k - M_k \sum_{j=0}^{k-1} | z^j | \\
&\geqq D_k | z |^k - M_k \frac{| z |^k - 1}{| z | - 1} \\
&\geqq \frac{| z |^k [D_k(| z | - 1) - M_k] + M_k}{| z | - 1} \\
&\geqq \frac{| z |^k}{| z | - 1} \left\{ [D_k | z | - (D_k + M_k)] + \frac{M_k}{| z |^k} \right\}
\end{aligned}$$

which is positive for $| z | > 1 + M_k/D_k = R_k$. Hence the magnitude of all zeros of $F(z)$ must be bounded by R, the larger of the numbers R' and R_k. The theorem follows at once by noting that $F(z)$ is analytic and that an analytic function has at most a finite number of zeros in any finite region.

A.3 *Proof of Theorem 2*

We now wish to derive an expression for the number of rhp zeros of $F(z)$. We choose a contour Γ varying along the imaginary axis from $-y$ to y (call this portion ω) where $y \geqq R_k$ of Theorem 1 and close it by a contour C outside the semicircle of radius R of Theorem 1. Let $F(z)$ and $\psi(z)$ be as defined in (15) and (16), respectively. We choose contour C (and increase y, if necessary) so that $| \psi(z) | < 1$ along C.

Let N be the number of zeros of $F(z)$ inside Γ and let

$$\Delta_\Gamma\ (F(z))$$

be the net change in arg $(F(z))$ along Γ. Then N, the number of zeros of $F(z)$ enclosed by Γ (assuming counterclockwise travel), is given by

$$N = \frac{1}{2\pi}\,\Delta_\Gamma(F(z)) = \frac{1}{2\pi}\,\Delta_\omega(F(z)) + \frac{1}{2\pi}\,\Delta_C(F(z))$$

$$= \frac{1}{2\pi}\,\Delta_\omega(F(z)) + \Delta_C(z^k) + \Delta_c(F_k(z)) + \Delta_C(1 + \psi(z))$$

since $F(z)$ has no zeros on Γ. Since $1 + \psi(z)$ does not wind around the origin (its real part being always positive), $\Delta_C(1 + \psi(z)) = 0$. Since $\Delta_C(z^k) = k/2$ we have thus proven Theorem 2.

A.4 *Sufficient conditions that Q_n does not wind around the origin*

Let

$$r_k = \min_{j=2,\cdots,m} r_j\ .$$

Then

$$\frac{e^{r_k x}}{a_{1n}}\,\mathrm{Re}\,[Q_n] = e^{r_k x} + \sum_{j=2}^{m} \frac{a_{jn}}{a_{1n}}\,e^{-(r_j-r_k)x}\cdot\cos r_j y$$

$$\frac{e^{r_k x}}{|\,a_{1n}\,|}\,|\,\mathrm{Re}\,[Q_n]\,| \geqq e^{r_k x} - \sum_{j=2}^{m}\left|\frac{a_{jn}}{a_{1n}}\right|.$$

Hence $|\,\mathrm{Re}\,[Q_n]\,| > 0$ for $x > 1/r_k \ln \alpha$ where

$$\alpha = \sum_{j=2}^{m}\left|\frac{a_{jn}}{a_{1n}}\right|.$$

Thus we need only consider a rectangle defined by $0 < x \leqq \ln \alpha/r_k$, $0 \leqq y < P$. [Note that any contour C will work if $\alpha < 1$ which is the case if assumption (11) is used.] $Q_n(e^{-z})$ does not wind around the origin for any contour with $x > \ln \alpha/r_k$, since Re $[Q_n]$ does not change sign. If $\sum_{j=1}^{m} a_{jn}^{-r_j x} \neq 0$ for $0 < x < \ln \alpha/r_k$ then Re $[Q_n]$ does not change sign for y any integer multiple of P, and $0 < x < \ln \alpha/r_k$. A suitable contour could then consist of a horizontal line at $y = KP$ from $x = 0$ to $x = \ln \alpha/r_k$, where K is an integer large enough so that $KP > R$ of Theorem 1. The rest of the contour in the first quadrant could be semicircular. The contour is completed in the fourth quadrant by the mirror image of the first quadrant. Since Re $[Q_n] \neq 0$ along this contour, $Q_n[e^{-z}]$ does not

wind around the origin. (Note that $a_{jn} > 0$, $j = 1, \cdots, m$ is sufficient to satisfy the conditions of this paragraph.) Furthermore, since Re $[Q_n]$ is even in y, $\Delta_C(Q_n(e^{-z})) = 0$ along the contour chosen.

This result can be extended to include the case where Re $[Q_n]$ has simple zeros on $y = KP$. In this case, use semicircular indentations around such points, in the direction to have Im $[Q_n] > 0$, *in both first and fourth quadrants.* (Hence the contour ceases to be symmetrical about the real axis.) Thus, along the deformed horizontal lines, the graph of $Q_n(e^{-z})$ remains in the upper-half plane. Along the semicircular portion it remains in either the right- or left-half plane. Hence no encirclements of the origin are possible and again $\Delta_C[Q_n(e^{-z})] = 0$.

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